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Invariant volumes of compact groups

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Abstract. Invariant volumes are calculated for all compact simple Lie groups. The measure is defined by means of the Riemannian metric, induced in the group by the Killing form, so the total normalisation is fixed. The coset space aspects are discussed.

1. Introduction

The concept of the invariant integral over the group manifold was introduced by H Weyl and is a powerful method in the theory of group representations (Weyl 1946, Helgason 1962, Warner 1972). However, in the spectral theory the absolute value of the group volume is usually an inessential constant which has scarcely attracted any special interest. The invariant measure for any group manifold is well known; but it is rather difficult to calculate the integral directly, because, in particular, the geometry of the manifold is on the whole not trivial.

Recently the calculation of the correctly normalised volumes became more urgent in view of the immense progress of the theory of gauge fields. As was shown by 't Hooft (1976) for the case of SU(2), the group volume (divided by a power of the coupling constant) appears in the expressions determining the effect of zero modes in the gauge field fluctuations around the instanton (pseudoparticle) solution. Other authors considered this subject for SU(3) (Bernard 1979, Geshkenbein and Ioffe 1979). Quite large symmetry groups are now discussed in particle physics (Gell-Mann *et al* 1978), and a general result may be of some interest. Even for SU(3) the correct number cannot be easily obtained directly; that given by Bernard (1979) is a factor of $\frac{4}{3}$ higher than ours.

The method used here, exploits the spectral expansion of the Green function for the quantal motion (or diffusion) on a group manifold. This expansion is written immediately in terms of the Weyl characters. On the other hand, the small-time (semiclassical) asymptotics of the Green function on any compact Riemannian manifold are determined by its most fundamental geometrical properties (Kac 1966, Molchanov 1975). The dominating term in the spectral sum is just the volume, divided by $(2\pi i\hbar t)^{n/2}$. So, starting from the exact Green function and expanding it near the singularity at $t = 0$, one gets the volume. The diffusion and the free quantal motion on compact Lie groups have been discussed in a number of works (Eskin 1963, Dowker 1970, 1971). Here I use the results and some notations from a paper by Marinov and Terentyev (1979).

In § 2 the terminology is explained and the notation is given. Section 3 states the results. The group volume is defined according to the Riemannian geometry of the group manifold. Relation to the Weyl measure is established in Appendix 1. The final expression is given in (19), while the relevant quantities are compiled in table 1. In § 4 we discuss some specific aspects of the matter. In particular, the most frequently used groups, orthogonal and unitary, are considered in view of their coset space structure. The large-N asymptotics of the invariant SU(N) volume are also given.

2. Preliminaries and notation

Let G be a compact simple Lie group; Z, its centre, A, the corresponding Lie algebra; W, its Weyl group. Suppose that G is simply connected (universal covering), $G_A = G/Z$ is the adjoint group. The basis elements of A are X_a, and

$$[X_a, X_b] = C^c_{ab} X_c, \tag{1}$$

where C^c_{ab} are the (real) structure constants. Elements of G are written by means of the exponent

$$g = \exp(\xi^a X_a), \tag{2}$$

where ξ^a are the (real) group parameters. The Killing matrix

$$G_{ab} = -C^c_{ad} C^d_{bc} \tag{3}$$

is positive definite in the case of a compact semi-simple group, and the inverse matrix exists $G_{ab}G^{bc} = \delta^c_a$. By means of a linear transformation in the algebra, G_{ab} may be reduced to a diagonal form $G_{ab} = \Lambda\delta_{ab}$, where Λ is a scale factor. In most applications certain values of $\Lambda \neq 1$ are, as a rule, suitable.

We use the following notations: r, the rank of A: n, the dimensionality; $p = (n-r)/2$, the number of positive roots; n(W), the order of W; n(Z), the order of Z. There is an ordered system of positive roots $\alpha^{(\nu)}$, $\nu = 1, \ldots, p$; among them are simple roots (a basis in the root space) $\boldsymbol{\gamma}^{(j)}$, $j = 1, \ldots, r$, and the highest root

$$\boldsymbol{\alpha}^{(1)} = \sum_{j=1}^{r} a_j \boldsymbol{\gamma}^{(j)}, \tag{4}$$

where a_j are positive integers. The scale factor Λ and a vector $\boldsymbol{\rho}$ are involved in some important formulae; they are expressed in terms of the roots

$$\Lambda = 2r^{-1} \sum_{\alpha>0} \boldsymbol{\alpha}^2, \qquad \boldsymbol{\rho} = \tfrac{1}{2} \sum_{\alpha>0} \boldsymbol{\alpha}; \tag{5}$$

the sums are over all positive roots. The Cartan matrix has elements

$$M_{jk} = 2(\boldsymbol{\gamma}^{(j)}\boldsymbol{\gamma}^{(k)})/(\boldsymbol{\gamma}^{(j)})^2. \tag{6}$$

that are integers 0, ±1, ±2, or -3. The following nice identities hold for irreducible root systems

$$n(\mathrm{Z}) = \det(M_{jk}), \tag{7}$$

$$n(\mathrm{W}) = n(\mathrm{Z}) r! \prod_{j=1}^{r} a_j, \tag{8}$$

$$n = 24\rho^2/\Lambda \equiv 3r(\Sigma\boldsymbol{\alpha})^2/\Sigma\boldsymbol{\alpha}^2. \tag{9}$$

These identities may be found in the literature (see also the remarks in § 4). The general proofs are rather sophisticated; one may be satisfied with a direct test of them for any simple Lie algebra.

Instead of the Cartesian coordinates ξ^a, defined by (2), polar coordinates $(\varphi_j\, \omega^\nu)$ $(\nu = 1, \ldots, 2p)$ are introduced on the group manifold

$$g = v^{-1}(\omega) h(\varphi) v(\omega), \tag{10}$$

where $h(\varphi) = \exp(\boldsymbol{\varphi H})$ are the elements of the maximal Abelian subgroup $\mathrm{H} \subset \mathrm{G}$ (the maximal torus), and H_j are basis elements of the Cartan subalgebra of A, $[H_j, H_k] = 0$. The coset space $\mathrm{G}/\mathrm{H} = \mathscr{V}$ is called the orbit space of the group.

There is a natural metric in the group manifold, as it is discussed by Eisenhart (1933), see also Marinov and Terentyev (1979), appendix 2. The line interval is given by

$$\mathrm{d}s^2 = g_{ab}(\xi)\, \mathrm{d}\xi^a\, \mathrm{d}\xi^b, \qquad g_{ab}(\xi) = B_a^p(\xi) B_b^q(\xi) G_{pq}, \tag{11}$$

where

$$B_a^b(\xi) = \int_0^1 A_a^b(\tau\xi)\, \mathrm{d}\tau, \tag{12}$$

and $A_a^b(\xi)$ is the adjoint representation matrix. Near the origin (unity of G) the metric is Euclidean, $g_{ab}(0) = G_{ab}$ because $B_a^b(0) = \delta_a^b$. It is easy to see that among the eigenvalues of the matrix B_a^b there are r unities and p complex conjugated pairs $2\exp[\pm \mathrm{i}(\boldsymbol{\alpha\varphi})/2] \sin \frac{1}{2}(\boldsymbol{\alpha\varphi})/(\boldsymbol{\alpha\varphi})$.

3. The invariant volume

Define the invariant volume of the group manifold by means of the Riemannian measure

$$V_{\mathrm{inv}}(\mathrm{G}) = \int_G |g(\xi)|^{1/2} \prod_{a=1}^{n} \mathrm{d}\xi^a, \tag{13}$$

where

$$|g(\xi)| = \det g_{ab}(\xi) = \det G_{ab} \prod_{\alpha>0} [2\sin\tfrac{1}{2}(\boldsymbol{\alpha\varphi})/(\boldsymbol{\alpha\varphi})]^4.$$

Using the polar coordinates, given in (10), and the fact that the Jacobian $\partial\xi/\partial(\boldsymbol{\varphi}, \boldsymbol{\omega})$ depends on $\boldsymbol{\varphi}$ just in the factor $\Pi_{\alpha>0}(\boldsymbol{\alpha\varphi})^2$(see Appendix 1), one may rewrite the measure in the Weyl form. Thus the invariant volume is

$$V_{\mathrm{inv}}(\mathrm{G}) = (\det G_{ab})^{1/2}\, V(\mathrm{H})\, V(\mathscr{V}), \tag{14}$$

where

$$V(\mathrm{H}) = \int_{\mathrm{H}} \prod_{\alpha>0} [2\sin\tfrac{1}{2}(\boldsymbol{\alpha\varphi})]^2 \prod \mathrm{d}\varphi = n(\mathrm{W})\, V_0(\mathrm{H}),$$

$$V(\mathscr{V}) = \int_{\mathscr{V}} \rho(\omega) \prod \mathrm{d}\omega^\nu, \qquad V_0(\mathrm{H}) = \int_{\mathrm{H}} \prod_{j=1}^{r} \mathrm{d}\varphi_j.$$

Here $V_0(\mathrm{H})$ is the Euclidean volume of the maximal torus, and the function $\rho(\omega)$ is given in (A1.6). The appearance of the order of the Weyl group $n(\mathrm{W})$ is here evident with the Weyl formula

$$\prod_{\alpha>0} 2\mathrm{i}\sin\tfrac{1}{2}(\boldsymbol{\alpha\varphi}) = \sum_{\sigma\in W} \epsilon_\sigma \exp \mathrm{i}(\sigma\rho, \varphi), \tag{15}$$

where the sum is over all elements of the Weyl group, $\epsilon_\sigma = \pm 1$ is the signature of the element σ.

First consider the adjoint group. The eigenvalues of the matrix $A_a^b(\xi)$ are r unities and p pairs $\exp[\pm \mathrm{i}(\alpha\varphi)]$. The elements of W, contained in V, permute the roots; it is appropriate to define the domain in the radial coordinates as $0 \leq (\boldsymbol{\alpha}^p\boldsymbol{\varphi}) \leq \ldots \leq (\boldsymbol{\alpha}^2\boldsymbol{\varphi}) \leq (\boldsymbol{\alpha}^1\boldsymbol{\varphi}) \leq \pi$, or equivalently, by means of $(r+1)$ inequalities

$$(\boldsymbol{\gamma}^{(j)}\boldsymbol{\varphi}) \geq 0, \qquad (\boldsymbol{\alpha}^1\boldsymbol{\varphi}) \leq \pi. \tag{16}$$

This is the Weyl alcove $\mathrm{H}_A = \mathrm{H}/\mathrm{Z}$. The volume of this domain is calculated immediately, for instance, by means of an induction in r,

$$V_0(\mathrm{H}_A) = \pi^r (\det(\boldsymbol{\gamma}^{(j)}\boldsymbol{\gamma}^{(k)}))^{-1/2}\left(r! \prod_{j=1}^{r} a_j\right)^{-1}. \tag{17}$$

A more concise formula for this volume may be also written, using (7) and (8). To find the volume of the universal covering group one has only to multiply (17) by $n(\mathrm{Z})$.

As for the orbit space, its volume is known from indirect considerations (see the next section),

$$V(\mathrm{V}) = (2\pi)^p \Big/ \prod_{\alpha>0} (\boldsymbol{\alpha\rho}), \tag{18}$$

see also (25).

The final expression is

$$\mathrm{V}_{\mathrm{inv}}(\mathrm{G}) = \Lambda^{n/2}(2\pi)^p \pi^r [n(\mathrm{Z})]^{3/2} / \Pi(\boldsymbol{\gamma}^2/2)^{1/2}\Pi(\boldsymbol{\alpha\rho}). \tag{19}$$

For all simple groups, the quantities involved are presented in table 1. A non-invariant volume of the ξ manifold, defined with the usual Euclidean volume element $\Pi\,\mathrm{d}\xi$ near $\xi = 0$, is given by $V_\xi(\mathrm{G}) = V_{\mathrm{inv}}(\mathrm{G})/(\det G_{ab})^{1/2}$. Of course, one is free to use an arbitrary basis in the Lie algebra (1); the volume is calculated respectively.

Table 1. Parameters characterising the geometry of simple Lie groups. Notations are given in § 2. The scaling dependent data in the last three columns are given for the canonical root systems tabulated in Bourbaki (1968).

Algebra	Group	n	p	$n(\mathrm{Z})$	$n(\mathrm{W})$	Λ	$\Pi(\boldsymbol{\gamma}^2/2)$	$\Pi(\boldsymbol{\alpha\rho})$
A_r	$\mathrm{SU}(r+1)$	$r(r+2)$	$\frac{1}{2}r(r+1)$	$r+1$	$(r+1)!$	$2(r+1)$	1	$\Pi_{s=1}^r s!$
B_r	$\mathrm{SO}(2r+1)$	$r(2r+1)$	r^2	2	$2^r r!$	$2(2r-1)$	$\frac{1}{2}$	$2^{-r}\Pi_{s=1}^r (2s-1)!$
C_r	$\mathrm{Sp}(2r)$	$r(2r+1)$	r^2	2	$2^r r!$	$4(r+1)$	2	$2^r\Pi_{s=1}^r(2s-1)!$
D_r	$\mathrm{SO}(2r)$	$r(2r-1)$	$r(r-1)$	4	$2^{r-1}r!$	$4(r-1)$	1	$2^{1-r}\Pi_{s=1}^{r-1}(2s)!$
	E_6	78	36	3	3.4!6!	24	1	4!5!7!8!11!
	E_7	133	63	2	4!4!7!	36	1	5!7!9!11!13!17!
	E_8	248	120	1	4!6!8!	60	1	7!11!13!17!19!23!29!
	F_4	52	24	1	2!4!4!	18	$\frac{1}{4}$	$2^{-12}5!7!11!$
	G_2	14	6	1	12	24	3	$3^3 5!$

4. Comments

4.1. Maximal torus

Any function on the group is periodical in the radial coordinates

$$f(\varphi+2\pi\nu)=f(\varphi), \tag{20}$$

where

$$\nu=\sum_{j=1}^{r} n_j\hat{\gamma}^{(j)}, \qquad \hat{\gamma}^{(j)}=2\gamma^{(j)}/(\gamma^{(j)})^2,$$

and n_j are integers. So the maximal torus for the group is inside a torus T, defined by $\varphi=\Sigma_j\psi_j\hat{\gamma}^{(j)}$, $-\pi<\psi_j\leqslant\pi$. Its Euclidean volume is

$$V_0(\mathrm{T})=(2\pi)^r\det(\hat{\gamma}_k^{(j)})\equiv(2\pi)^r(\det M_{jk})^{1/2}/\Pi(\gamma^2/2)^{1/2}. \tag{21}$$

Reflection symmetries of the group manifold result in further reduction of the essential domain in the φ space. With (8) and (17) one has

$$V_0(\mathrm{T})/V_0(\mathrm{H}_A)=2^r n(\mathrm{W}). \tag{22}$$

Actually, this result could be obtained immediately from the geometry, and then used to prove (8). As for the maximal torus of the universal covering group, its shape is not as simple as that of the Weyl alcove (16).

4.2. Orbit space

The volume of the orbit space (18) is obtainable from the general results of the spectral analysis on the Lie groups (Warner 1972 (vol 2, p 68), Flensted–Jensen 1978). Perhaps, the clearest deduction is that exploiting properties of the Green function for free quantal motion or diffusion on the group manifold (Eskin 1963, Dowker 1971, Marinov and Terentyev 1979). The Green function may be represented by the spectral expansion, as well as semiclassically, i.e. summing up the contributions from the classical trajectories. Suppose the Hamiltonian is given by the Laplace operator Δ on the group, corresponding to the metric (11), $\hat{H}=-\frac{1}{2}\hbar^2\Delta$, then the time-dependent Green function is

$$C\sum_{l} d_l\chi_l(\varphi)\exp(-\tfrac{1}{2}\mathrm{i}\hbar\epsilon(l)t)$$
$$=(2\pi\mathrm{i}\hbar t)^{-n/2}\sum_{\nu}\frac{(\alpha\Phi_\nu)}{2\sin\frac{1}{2}(\alpha\Phi_\nu)}\exp\left(\frac{\mathrm{i}}{2\hbar t}G_{jk}\Phi_\nu^j\Phi_\nu^k+\mathrm{i}\hbar nt/48\right). \tag{23}$$

Here l is the representation highest weight, d_l the dimensionality, $\chi_l(\varphi)$ the Weyl character, $\epsilon(l)=\Lambda^{-1}(l^2+2\rho l)$ is the eigenvalue of the invariant Casimir operator $G^{ab}X_aX_b$; $\Phi_\nu=\varphi+2\pi\nu$, as in (20). Evidently, the semiclassical sum is defined on the torus T, so

$$C^{-1}=(\det G_{ab})^{1/2}V_0(\mathrm{T})V(\mathcal{V}). \tag{24}$$

The constant C is calculated explicitly, comparing both the expansions, say, at small t and φ. Then using (21) one gets (18).

Having calculated (18) for all simple groups, we observed that

$$\prod_{\alpha>0} (\alpha\rho) = \prod_{\alpha>0} (\alpha^2/2) \prod_{j=1}^{r} m_j! \tag{25}$$

where m_j are indices of the Weyl group (exposants, see Bourbaki 1968). Note that $m_1 = 1$, $\Sigma_{j=1}^r m_j = p$, and $\Pi_{j=1}^r (m_j + 1) = n(\mathrm{W})$. Probably (25) as well as (9) has a direct geometrical meaning and may be proven in a general way.

4.3. *Orthogonal groups* SO(N)

Realisation of the group generators by means of the first-order differential operators in a real N-dimensional Euclidean space with coordinates x_μ,

$$X_a \to L_{\mu\nu} = x_\mu \partial/\partial x_\nu - x_\nu \partial/\partial x_\mu, \qquad 1 \leqslant \mu < \nu \leqslant N,$$

results in a diagonal Killing matrix (3), given by

$$G_{\mu\nu,\mu'\nu'} = 2(N-2)(\delta_{\mu\mu'}\delta_{\nu\nu'} - \delta_{\mu\nu'}\delta_{\nu\mu'}). \tag{26}$$

So the scale factor Λ is the same as that for the canonical root systems, given in table 1. Writing the group elements in terms of the rotation angles, $g(\omega) = \exp(\Sigma_{\mu<\nu}\omega_{\mu\nu}L_{\mu\nu})$, one gets the normalised volume of the ω manifold for the adjoint groups, omitting in (19) the factors $\Lambda^{n/2}$ and $n(\mathrm{Z})$,

$$\begin{aligned} V_\omega(\mathrm{SO}(2r+1)_A) &= 2(2\pi)^{r(r+1)} \Big/ \prod_{s=1}^{r-1} (2s+1)!, \\ V_\omega(\mathrm{SO}(2r)_A) &= (2\pi)^{r^2} \Big/ \prod_{s=1}^{r-1} (2s)!. \end{aligned} \tag{27}$$

In the odd case, the group of space rotations is isomorphic to the adjoint group, while in the even case it has a centre Z_2 (the inversion of all coordinate axes is a rotation commuting with other group elements). Introducing a special notation† for the vector rotation group $\mathrm{SO}(N)_v$ (the group of real orthogonal N×N matrices), one may write $\mathrm{SO}(2r+1)_A = \mathrm{SO}(2r+1)_v$, $\mathrm{SO}(2r)_A = \mathrm{SO}(2r)_v/Z_2 = \mathrm{SO}(2r)/\mathrm{Z}$, $\mathrm{SO}(N)_v = \mathrm{SO}(N)/Z_2$. Here the centre Z in the even case is $Z_2 \times Z_2$ for $r = 2m$ and Z_4 for $r = 2m+1$. Consequently,

$$\begin{aligned} V_\omega(\mathrm{SO}(2r+1)_v)/V_\omega(\mathrm{SO}(2r)_v) &= \tfrac{1}{2}V(S_{2r}), \\ V_\omega(\mathrm{SO}(2r)_v)/V_\omega(\mathrm{SO}(2r-1)_v) &= V(S_{2r-1}), \end{aligned} \tag{28}$$

where $V(S_N) = 2\pi^k/\Gamma(k)$ is the volume of the unit N-dimensional sphere S_N, $k = \frac{1}{2}(N+1)$. This is an indication to the topological properties of the coset space, $\mathrm{SO}(2r+1)_v/\mathrm{SO}(2r)_v = S_{2r}/Z_2$. The simplest example is given by $r = 1$; the coset space $\mathrm{SO}(3)_v/\mathrm{SO}(2)$ is the sphere with identified opposite points, because the rotation by an angle φ is the same as the rotation by $2\pi - \varphi$ with the axis in the opposite direction. The factor of $\frac{1}{2}$ in (28) is the reason for our disagreement with Gilmore (1974).

To construct the universal covering groups $\mathrm{SO}(2r+1)$ and $\mathrm{SO}(2r+2)$ and their centres explicitly, one must use the 2^r-dimensional spinor space. The group generators are written then in terms of the $2r$-dimensional Clifford algebra. The consideration is simple, but perhaps not to be followed along the present path.

† Some authors use instead a special notation *Spin*(N) for the universal covering of SO(N).

It is remarkable that (27) is true even at $r=1$. Note also that $V_\omega(SO(4)_A)=\frac{1}{8}[V_\omega(SO(3)_A)]^2$, because a rescaling of the rotation angles is necessary in carrying out the isomorphic mapping $SO(4)=SO(3)\times SO(3)$. Meanwhile, $V_{inv}(SO(4))=[V_{inv}(SO(3))]^2$.

4.4. *Unitary groups* SU(N)

Elements of SU(N) are represented with unitary unimodular $N\times N$ matrices, sometimes written as $\exp(i\lambda_a\xi^a)$, where λ_a, $a=1,\ldots,N^2-1$, are Hermitian, traceless matrices, orthonormal in the sense

$$\mathrm{Tr}(\lambda_a\lambda_b)=2\delta_{ab}, \tag{29}$$

just as the Pauli matrices in the case $N=2$. The scale of the group parameters is fixed by (29), corresponding to $\Lambda=4N$, while for the canonical root system (table 1) $\Lambda=2N$. To see this, one may use the general formula, valid for any matrix representation of the Lie algebra, $X\to X^1$,

$$\mathrm{Tr}(X_a^lX_b^l)=-G_{ab}\epsilon(l)d_l/n. \tag{30}$$

(The notations are the same as in (23).) For the fundamental representation $d=N$ and $\epsilon=\frac{1}{2}(1-N^{-2})$ (cf. Marinov and Terentyev 1979). Thus the volume of the ξ manifold is

$$V_\xi(SU(N))=2^{-\frac{1}{2}(N-1)}\pi^{\frac{1}{2}(N-1)(N+2)}N^{3/2}\Big/\prod_{s=1}^{N-1}s!. \tag{31}$$

In particular, $V_\xi(SU(2))=2\pi^2=V(S_3)$, $V_\xi(SU(3))=\frac{3}{4}\pi^5\sqrt{3}$. Note that $V_{inv}(SU(2))=V_{inv}(SO(3))$, while according to (27), $V_\omega(SO(3))=2V_\omega(SO(3)_A)=16\pi^2$. The factor of eight is due to the difference in the scale, as $L_{12}\to\frac{1}{2}\lambda_3$, etc.

Consider now the coset space $SU(N+1)/SU(N)$, which is in a sense locally isomorphic to the sphere S_{2N+1}. However, there is still a problem of a consistent scaling. Write the ratio of the group volumes as follows

$$V_\xi(SU(N+1))/V_\xi(SU(N))=\tfrac{1}{2}V(S_{2N+1})[(N+1)/2N]^{1/2}(1+N^{-1}). \tag{32}$$

The factor of $\frac{1}{2}$ for even N is of the same origin as in (29); the inversion of all axes is an element of the group, so the coset space has the topology of S_{2N+1}/Z_2. The third factor is due to the structure of the $(N+1)$-dimensional matrix, commuting with elements of the subgroup SU(N), $\lambda=[2N/(N+1)]^{1/2}\,\mathrm{diag}(-N^{-1},\ldots,-N^{-1},1)$, cf. Bernard (1979). Finally, the last factor, absent for embedding of the adjoint groups, is the ratio of the centre orders.

The present discussion shows that a calculation of the group volumes, based on the coset space metric without inspection of the whole geometry, used by Gilmore (1974) and Bernard (1979), is rather subtle. Our direct method provides us with results in disagreement with each of the mentioned authors.

4.5. *Asymptotics*

It is remarkable that at large r the leading factors in the invariant volumes (19) cancel. To estimate the asymptotics one has to derive an expansion for the product of factorials, present in (25). Using the expansion suggested in Appendix 2, one gets for SU(N) at $N\to\infty$

$$\ln V_{\rm inv}(\mathrm{SU}(N))=\tfrac{1}{2}(N^2-1)(\ln(4\pi)+3/2)-(N-1)\ln 2+\tfrac{13}{12}\ln N+\tfrac{5}{12}\ln 2-c+\mathrm{O}(N^{-1}), \qquad (33)$$

where the constant c is given in (A2.2).

Appendix 1. Jacobian for polar coordinates on the group manifold

Group elements may be represented in Cartesian coordinates (2) or in polar coordinates, as in (10). The relation between both the systems is

$$\xi^a=\sum_j A^a_j(\omega)\varphi_j, \qquad (\mathrm{A}1.1)$$

where $A^a_b(\omega)$ is the adjoint representation matrix for the group element $v(\omega)$. The Jacobi matrix J^a_A, $A=j$ or ν, has the elements

$$J^a_j\equiv\partial\xi^a/\partial\varphi_j=A^a_j(\omega),$$
$$J^a_\nu\equiv\partial\xi^a/\partial\omega^\nu=\sum_j\varphi_j\,\partial A^a_j/\partial\omega^\nu. \qquad (\mathrm{A}1.2)$$

The Cartesian coordinates may be established for the element v as well, $v(\omega)=\exp(X_a\eta^a)$, while η^a are functions of ω. Having in mind the known formula for a derivative of an exponent matrix, as well as the definition of the adjoint group representation, one may write

$$\partial A^a_j/\partial\omega^\nu=\partial\eta^b/\partial\omega^\nu B^c_b(\eta)C^d_{cj}A^a_d(\omega). \qquad (\mathrm{A}1.3)$$

It is appropriate to calculate the auxiliary matrix

$$\tilde G_{AB}(\xi)=J^a_AJ^b_BG_{ab}, \qquad (\mathrm{A}1.4)$$

then $\det\tilde G_{AB}=J^2\det G_{ab}$, where $J=\det J^a_A$ is the Jacobian. Using the invariance property of the Killing form, $A^a_c(\omega)A^b_d(\omega)G_{ab}=G_{cd}$, the definition of the root vectors in terms of the structure constants in the canonical basis, $C^{\alpha}_{j\beta}=\mathrm{i}\alpha_j\delta(\boldsymbol{\alpha},\boldsymbol{\beta})$, and the explicit form of G_{ab} in the canonical basis, after some algebra one obtains

$$\det\tilde G_{AB}=\prod_{\boldsymbol{\alpha}>0}(\boldsymbol{\alpha\varphi})^4\det\tilde g_{\mu\nu}(\omega). \qquad (\mathrm{A}1.5)$$

Here $\tilde g_{\mu\nu}=g_{ab}(\eta)\,\partial\eta^a/\partial\omega^\mu\,\partial\eta^b/\partial\omega^\nu$ represents the metric induced in the orbit space $\mathscr{V}$ by the original metric (11). Thus the Jacobian is

$$J=\prod_{\boldsymbol{\alpha}>0}(\boldsymbol{\alpha\varphi})^2\rho(\omega), \qquad \rho(\omega)=[\det\tilde g_{\mu\nu}(\omega)/\det G_{ab}]^{1/2}. \qquad (\mathrm{A}1.6)$$

The choice of the 'angular' coordinates ω is not essential at present. For instance, they may be related to the canonical basis, so that $v(\omega)=\exp(\Sigma_\alpha E_\alpha\omega^\alpha)$.

Appendix 2. Product of factorials in asymptotics

Exploiting the method used in obtaining the integral representation of $\ln \Gamma(z)$ and the asymptotics (Whittaker and Watson 1952), one gets, extracting the singularity at large r,

$$\ln \prod_{s=1}^{r} s! = \tfrac{1}{2}r(r+2)\ln(r+2) + \tfrac{5}{12}\ln(\tfrac{1}{2}r+1) - r(\tfrac{3}{4}r + 2 - \tfrac{1}{2}\ln 2\pi) + c - \int_0^\infty \varphi(t)\exp[-(r+2)t]\,\mathrm{d}t, \tag{A2.1}$$

where

$$\varphi(t) = t^{-1}[(1-\mathrm{e}^{-t})^{-2} - t^{-2} - t^{-1} - \tfrac{5}{12}] \sim \tfrac{1}{12} + \mathrm{O}(t),$$
$$c = \int_0^\infty \varphi(t)\,\mathrm{e}^{-2t}\,\mathrm{d}t. \tag{A2.2}$$

The integral in (A2.1) behaves as $\mathrm{O}(r^{-1})$ at $r \to \infty$; expanding $\varphi(t)$ in powers of t, one may get an analogue of the Stirling expansion for the product of factorials.

Additional note

After this paper was submitted for publication, two relevant mathematical works were shown to the authors. The more recent one is that by McDonald (1980). Based on the fact that the group manifold has the same cohomology, up to the torsion, as a product of spheres of dimensions $2m_j+1 (j=1,\ldots,r)$, the author extracts the product of the sphere volumes from the group volume, using (25). This notable identity was for a time known by Steinberg. Steinberg's deduction (published by Harder 1971, p 454) is formally simple, but it stems from an algebraic property of the basic invariants of the Lie algebra, the numbers (m_j+1) being the degrees of the invariant polynomials (Freudenthal and deVries 1969). A formula for the group volume containing a multiple integral was given also by Cahn (1974). By means of a substitution, this integral may be reduced to the integral over the Lie algebra, that was explicitly calculated by McDonald (1980). In fact, both the authors exploited the same normalisation property of the zeta function (a physicist would prefer the name Green function) on the group manifold, so our starting points are equivalent. However, neither of the authors mentioned presents an ultimately explicit expression, like that of equation (19).

References

Bernard C 1979 *Phys. Rev.* D **19** 3013–9
Bourbaki N 1968 *Eléments de Mathmatique, Fasc. 34. Groupes et algèbres de Lie* Ch 4–6 (Paris: Hermann)
Cahn R S 1974 *Proc. Am. Math. Soc.* **46** 247–9
Dowker J S 1970 *J. Phys. A: Math. Gen.* **3** 451
—— 1971 *Ann. Phys., NY* **62** 361–82
Eisenhart L P 1933 *Continuous Groups of Transformations* (Princeton: University Press)
Eskin L D 1963, in *Pamyati Chebotareva* (in Russian) (Kazan: Kazan State University) pp 113–32
Flensted-Jensen M 1978 *J. Funct. Anal.* **30** 106–46
Freudenthal H and de Vries H 1969 *Linear Lie Groups* (New York, London: Academic Press) § 77
Gell-Mann M, Ramond P and Slansky R 1978 *Rev. Mod. Phys.* **50** 721–44

Geshkenbein B V and Ioffe B L 1979 *The role of instantons in generation of mesonic mass spectrum* Preprint ITEP-82 (Moscow: ITEP)
—— 1980 *Nucl. Phys.* B **166** 340–67
Gilmore R 1974 *Lie groups, Lie Algebras and Some of Their Applications* (New York, London, Toronto: Wiley)
Harder G 1971 *Ann. Scient. École Norm. Sup.* **4** 409–55
Helgason S 1962 Differential Geometry and Symmetric Spaces (New York, London: Academic Press)
't Hooft G 1976 *Phys. Rev.* D **14** 3432
—— 1978 *Phys. Rev.* D **18** 2199(E)
Kac M 1966 *Amer. Math. Monthly* **73** 1–23
McDonald I G 1980 *Inv. math.* **56** 93–5
Marinov M S and Terentyev M V 1979 *Fortschr. Phys.* **27** 511–45
Molchanov S A 1975 *Usp. Mat. Nauk* (USSR) **30** (1) 3–59
Warner G 1972 *Harmonic Analysis on Semi-Simple Lie Groups*, 2 vols (New York, Heidelberg, Berlin: Springer)
Weyl H 1946 *The Classical Groups* (Princeton: University Press)
Whittaker E T and Watson G N 1952 *A Course of Modern Analysis* (Cambridge: University Press) Ch 12